1 Generalised mean

1.1 General form

$$M_p(a_1, ..., a_n) = (\frac{1}{n}\sum_{i=1}^{n} a_1^p)^{\frac{1}{p}}$$

1.2 Arithmetic mean

$$\begin{aligned} M_1(a_1, ..., a_n) &= (\frac{1}{n}\sum_{i=1}^{n} a_i^1)^{\frac{1}{1}} \\ &= (\frac{1}{n}\sum_{i=1}^{n} a_i) \\ \bar{a} &= \frac{a_1 + a_2 + ... + a_{n-1} + a_n}{n} \\ \bar{a}n &= a_1 + a_2 + ... + a_{n-1} + a_n \end{aligned}$$

1.3 Geometric mean

$$\begin{aligned} M_0(a_1, ..., a_n) &= \lim_{p \to 0}(\frac{1}{n}\sum_{i=1}^{n} a_i^p)^{\frac{1}{p}} \\ &= \sqrt[n]{a_1 a_2 \cdot ... \cdot a_{n-1} a_n} = \bar{g} \\ \bar{g} &= (a_1 a_2 \cdot ... \cdot a_{n-1} a_n)^{\frac{1}{n}} \\ \bar{g}^n &= a_1 a_2 \cdot ... \cdot a_{n-1} a_n \end{aligned}$$

1.4 Harmonic mean

$$\begin{aligned} M_{-1}(a_1, ..., a_n) &= (\frac{1}{n}\sum_{i=1}^{n} a_i^{-1})^{-1} \\ &= \frac{1}{\frac{1}{n}(\frac{1}{a_1} + ... + \frac{1}{a_n})} \\ \bar{h} &= \frac{n}{\frac{1}{a_1} + ... + \frac{1}{a_n}} \\ \frac{n}{\bar{h}} &= \frac{1}{a_1} + ... + \frac{1}{a_n} \end{aligned}$$

2 L^p norm

2.1 General form

$$||x||_p = (\sum_{i=1}^{n} |x_i^p|)^{\frac{1}{p}}$$

2.2 L1

For L1, the distance is equivalent to the sum of absolute distances:

$$||x||_1 = (\sum_{i=1}^{n} |x_i|)$$

2.3 L2 or Euclidean norm

$$\begin{aligned}||x||_2 &= (\sum_{i=1}^{n} |x_i^2|)^{\frac{1}{2}} \\ &= \sqrt[2]{x_1^2 + ... + x_n^2}\end{aligned}$$

2.4 L-infinity

For L-infinity, the distance is equal to the maximum distance observed:

$$||x||_\infty = \max_i |x_i|$$

3 Mean of a function

Since the defining property of of the arithmetic mean of a sequence is that it added to itself n times equals the sum of the sequence, $n\bar{a} = a_1 + a_2 + ... + a_{n-1} + a_n$, similarly the average value of a function f over the interval $[a, b]$ is

$$\int_a^b \bar{f} dx = \int_a^b f(x) dx$$

But now the LHS, an integral of a constant, is

$$\begin{aligned}\int_a^b \bar{f} dx &= \bar{f}x\Big|_a^b \\ &= \bar{f}b - \bar{f}a \\ &= (b - a)\bar{f}\end{aligned}$$

Thus

$$\begin{aligned}\int_a^b \bar{f} dx = (b - a)\bar{f} &= \int_a^b f(x) dx \\ \iff \bar{f} &= \frac{1}{b - a} \int_a^b f(x) dx\end{aligned}$$

4 Time averages and ensemble averages

4.1 Finite-time average

Finite-time average is equivalent to the mean value of the system within interval $[t, t + \Delta t]$:

$$\bar{A}_{\Delta t} = \frac{1}{\Delta t} \int_{t}^{t+\Delta t} A(s) ds$$

4.2 Time average

The actual time average is when the interval considers the whole time space for the system:

$$\bar{A} = \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \int_{t}^{t+\Delta t} A(s) ds$$

4.3 Finite-ensemble average

$$\ddot{A}_t = \frac{1}{N} \sum_{i}^{N} A_i(t)$$

4.4 Ensemble average

$$\ddot{A} = \lim_{N \to \infty} \frac{1}{N} \sum_{i}^{N} A_i(t)$$

$$\ddot{A} = \int_{-\infty}^{+\infty} P(A(s)) A(s) ds$$

5 Ergodicity