Workshop 2: Definite integrals and Riemann sums

Problem 1

Part 1.1

Question: Write using the sigma ($\sum$) -notation the sum of the first n+1 non negative (0 is also not negative) integers. Use the variable k to denote an integer. You will obtain a rather simple expression but pay attention to details such as summation bounds. Denote the sum as S(n,k).

Solution: Since we are after the $n+1$ first integers and since 0 is allowed, let's start the summation from 0:

$$S(n,k) = \sum_{k=0}^{n} k$$

This way, say we want the first $n+1 = 4$ integers where $n = 3$. We have

$$S(3,k) = \sum_{k=0}^{3} k = 0 + 1 + 2 + 3 = 6$$

Part 1.2

Question: Show that $S(n,k) = S(n, n-k)$. Hint: $S(n, n-k)$ is the sum in which k is replaced by $n-k$. The right hand side of the equation can be seen as some sort of reordering of the terms of the left hand side.

Solution: Let's do a direct proof:

$$\begin{aligned} S(n,k) = \sum_{k=0}^{n} k &= 0 + 1 + 2 + ... + (n-1) + n \\ &= n + (n-1) + ... + 2 + 1 + 0 \\ &= \sum_{k=0}^{n} (n-k) \\ &= S(n, n-k) \end{aligned}$$

So $S(n, n-k)$ simply sums the terms in the reverse order.

Part 1.3

Question: Show that $S(0, k) = 0$ and $S(n+1, k) = S(n, k) + n + 1$. Hint: This can be shown without induction by using the definitions of the expressions of both sides of the equation.

Solution: Direct proofs are suitable for both. First, $S(0, k) = 0$:

$$S(0, k) = \sum_{k=0}^{0} k = 0$$

As for $S(n+1, k) = S(n, k) + n + 1$,

$$\begin{aligned} S(n+1, k) &= \sum_{k=0}^{n+1} k \\ &= 0 + 1 + 2 + ... + (n-1) + n + (n+1) \\ &= \big[0 + 1 + 2 + ... + (n-1) + n\big] + n + 1 \\ &= \Big[\sum_{k=0}^{n} k\Big] + n + 1 \\ &= S(n, k) + n + 1 \end{aligned}$$

Part 1.4

Question: Argue that the sum $S(n, k)$ is a polynomial in the variable n of degree ≤ 2. Hint: Use part 2. to argue that it is a sum of $n+1$ polynomials of degree ≤ 1. At this point it is not necessary to find an expression of the polynomial. A heuristic argument is enough.

Solution:

Part 1.5

Question: Denote the polynomial $S(n, k)$ of degree ≤ 2 as $P(n) = a_2 n^2 + a_1 n + a_0$. Show that the coefficient $a_0 = 0$. Hint: look at the first result from part 3.

Solution: We have

$$S(n, k) = P(n) = a_2 n^2 + a_1 n + a_0$$

then, with this definition, for $n = 0$ we obtain a polynomial

$$\begin{aligned} S(0, k) = P(0) = a_2(0)^2 + a_1(0) + a_0 &= 0 \\ \iff a_0 &= 0 \end{aligned}$$

Part 1.6

Question: Solve the other coefficients of $P(n)$. Hint: Observe that from the second result of part 3 we have that $P(n+1) = P(n) + n + 1$. Rewrite this equation using the polynomial from part 5. Expand the products in the left hand side and collect like terms in both sides separately. You should obtain an equation between two polynomials. Two polynomials are equal if and only if they have the same coefficients. This will lead to a system of 3 linear equations. Solving it will yield the coefficients.

Solution: We have

$$\begin{aligned} P(n+1) &= P(n) + n + 1 \\ \iff a_2(n+1)^2 + a_1(n+1) + a_0 &= a_2n^2 + a_1n + a_0 + n + 1 \\ a_2(n^2 + 2n + 1) + a_1n + a_1 + a_0 &= a_2n^2 + a_1n + a_0 + n + 1 \\ a_2n^2 + 2a_2n + a_2 + a_1n + a_1 + a_0 &= a_2n^2 + a_1n + a_0 + n + 1 \\ a_2n^2 + (2a_2 + a_1)n + (a_2 + a_1 + a_0) &= a_2n^2 + (a_1 + 1)n + (a_0 + 1) \end{aligned}$$

Because two polynomials are equal iff they have equal coefficients, we obtain the following system of equations by setting the coefficients equal

$$\begin{aligned} a_2 &= a_2 \\ 2a_2 + a_1 &= a_1 + 1 \quad &\implies a_2 = 1/2 \\ a_2 + a_1 + a_0 &= a_0 + 1 \quad &\implies a_1 = 1/2 \end{aligned}$$

Part 1.7

Question: Write the sum from part 1 as a polynomial.

Solution: Collecting together all that we found out, we have

$$S(n,k) = \sum_{k=0}^{n} k = \frac{1}{2}n^2 + \frac{1}{2}n + 0 = P(n)$$

Problem 2

Question: Compute $\int_0^1 2x dx$ using the definition of 'the definite integral as a limit of Riemann sum'.

Solution: We can partition the interval $[0, 1]$ into n subintervals of equal length:

$$0, \frac{1}{n}, \frac{2}{n}, ..., \frac{k-1}{n}, \frac{k}{n}, ..., 1$$

From this we notice that each subinterval is of length

$$\frac{k}{n} - \frac{k-1}{n}$$

This can be used as the base when computing an area of a rectangle. Now we need only the height. The function we are integrating is $2x$. That is enough to compute the height. The height of the graph at the left boundary of each interval is

$$2\left[\frac{k-1}{n}\right]$$

and the height at the right boundary of each interval is

$$2\left[\frac{k}{n}\right]$$

Since $D_x[2x] = 2$, that is, we have a positive slope, the area computed using the height at left boundaries systematically underestimates the true area, while the area computed using the height at right boundaries systematically overestimates the area, we can get both lower and upper bounds for the true area. The lower bound is

$$\overline{S_n} = \sum_{k=1}^{n} 2\left[\frac{k-1}{n}\right]\frac{1}{n}$$

while the upper bound is

$$\underline{S_n} = \sum_{k=1}^{n} 2\left[\frac{k}{n}\right]\frac{1}{n}$$

Now we have

$$\underline{S_n} = \sum_{k=1}^{n} 2\left[\frac{k-1}{n}\right]\frac{1}{n} \leq A \leq \sum_{k=1}^{n} 2\left[\frac{k}{n}\right]\frac{1}{n} = \overline{S_n}$$

We can use to our advantage the fact that $\sum_{k=0}^{n} k = \frac{n(1+n)}{2}$ and rearrange the terms to give

$$
\begin{aligned}
&\sum_{k=1}^{n} 2\Big[\frac{k-1}{n}\Big]\frac{1}{n} \leq A \leq \sum_{k=1}^{n} 2\Big[\frac{k}{n}\Big]\frac{1}{n} \\
&\iff 2\sum_{k=1}^{n} \frac{k-1}{n^2} \leq A \leq 2\sum_{k=1}^{n} \frac{k}{n^2} \\
&\iff \frac{2}{n^2}\sum_{k=1}^{n} k-1 \leq A \leq \frac{2}{n^2}\sum_{k=1}^{n} k \\
&\iff \frac{2}{n^2}\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 \leq A \leq \frac{2}{n^2}\sum_{k=1}^{n} k \\
&\iff \frac{2}{n^2}\sum_{k=1}^{n} k - \frac{2}{n^2}\sum_{k=1}^{n} 1 \leq A \leq \frac{2}{n^2}\sum_{k=1}^{n} k \\
&\iff \frac{2}{n^2}\Big[\frac{n(1+n)}{2}\Big] - \frac{2}{n^2}\cdot n \leq A \leq \frac{2}{n^2}\Big[\frac{n(1+n)}{2}\Big] \\
&\iff \frac{1+n}{n} - \frac{2}{n} \leq A \leq \frac{1+n}{n} \\
&\iff \frac{n+n^2-2n}{n^2} \leq A \leq \frac{1+n}{n} \\
&\iff \frac{n^2-n}{n^2} \leq A \leq \frac{1+n}{n}
\end{aligned}
$$

We can now take the limit of both sides as n goes to ∞:

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{n^2-n}{n^2} \leq A \leq \lim_{n\to\infty} \frac{1+n}{n} \\
&\iff \lim_{n\to\infty} \frac{1-\frac{1}{n}}{1} \leq A \leq \lim_{n\to\infty} \frac{\frac{1}{n}+1}{1} \\
&\iff 1 \leq A \leq 1
\end{aligned}
$$

Thus we have the area of the initial integral:

$$\int_0^1 2x\,dx = 1$$

Problem 3

Question: Express the limit $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n+k}{n^2}$ as a definite integral. Compute this integral by interpreting it as the area of a known domain.

Solution: We can manipulate the above sum into a Riemann sum form:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n+k}{n^2}$$
$$\iff \lim_{n\to\infty}\sum_{k=1}^{n}\frac{n+k}{n}\frac{1}{n}$$
$$\iff \lim_{n\to\infty}\sum_{k=1}^{n}\Big(\frac{k}{n}+1\Big)\frac{1}{n}$$

Now this is a right Riemann sum for a function $f(x) = x + 1$. We have

$$\lim_{n\to\infty}\sum_{k=1}^{n}\Big(\frac{k}{n}+1\Big)\frac{1}{n} = \int_0^1 (x+1)\,dx = \Big[\frac{1}{2}x^2+x\Big]_0^1 = \frac{1}{2}+1 = \frac{3}{2}$$

Problem 4

Question: Find

$$\lim_{n\to\infty}\frac{\sqrt{1}+\sqrt{2}+...+\sqrt{n}}{n\sqrt{n}}$$

Solution: We can first rewrite the term we are taking a limit of as

$$\frac{\sqrt{1}+\sqrt{2}+...+\sqrt{n}}{n\sqrt{n}} = \Big[\frac{\sqrt{1}+\sqrt{2}+...+\sqrt{n}}{\sqrt{n}}\Big]\frac{1}{n}$$

The above, up to a finite n, can be written as a summation:

$$\sum_{k=1}^{n}\frac{\sqrt{k}}{\sqrt{n}}\frac{1}{n} = \sum_{k=1}^{n}\sqrt{\frac{k}{n}}\frac{1}{n}$$

Now we can notice a Riemann sum in the above form: we can interpret $1/n$ as the length of an interval and $\sqrt{k/n}$ as the value of a function at the right boundary. We can now take the limit and interpret it as an integral:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}\frac{1}{n} = \int_0^1 \sqrt{x}\,dx = \Big[\frac{2}{3}(x)^{\frac{3}{2}}\Big]_0^1 = \frac{2}{3}$$