Workshop 3: The Fundamental Theorem of Calculus

Problem 1

Question: Consider the function F that is defined as

$$F(x) = \int_0^x e^{-\sin t}\,dt$$

Show that the function F is increasing.

Solution: A function is increasing if its derivative is always non-negative. So let's take the derivative of $F(x)$:

$$F'(x) = e^{-\sin x} = \frac{1}{e^{\sin x}}$$

For the sine function

$$-1 \leq \sin x \leq 1$$

and so for the derivative

$$0 < \frac{1}{e} \leq \frac{1}{e^{\sin x}} \leq \frac{1}{e^{-1}} = e$$

Since the derivative is always non-negative, the $F(x)$ in question is increasing.

Problem 2

Question: Let the function F be defined by

$$F(x) = \int_x^{2x} \sin\left(t^2\right)dt$$

Compute $F'(x)$.

Solution: We begin by noticing that the integral can be represented as

$$F(x) = \int_x^{2x} \sin(t^2)\, dt = \int_0^{2x} \sin(t^2)\, dt - \int_0^{x} \sin(t^2)\, dt = F_1(g(x)) - F_2(x)$$

For some $h \neq 0$ we have

$$\begin{aligned}
F'(x) &= \Big[F_1[g(x+h)] - F_1[g(x)]\Big] - \Big[F_2(x+h) - F_2(x)\Big] \\
&= \left[\int_0^{2x+2h} \sin(t^2)\, dt - \int_0^{2x} \sin(t^2)\, dt\right] - \left[\int_0^{x+h} \sin(t^2)\, dt - \int_0^{x} \sin(t^2)\, dt\right] \\
&= \left[\int_{2x}^{2x+2h} \sin(t^2)\, dt\right] - \left[\int_x^{x+h} \sin(t^2)\, dt\right]
\end{aligned}$$

Applying the intermediate value theorem for the two integrals separately, we have

$$\begin{aligned}
F_1[g(x+h)] - F_1[g(x)] &= \int_{2x}^{2x+2h} \sin(t^2)\, dt = f_1(g(z))h \\
\iff \frac{F_1[g(x+h)] - F_1[g(x)]}{h} &= \frac{\int_{2x}^{2x+2h} \sin(t^2)\, dt}{h} = f_1(g(z))
\end{aligned}$$

For some z where $x < z < x+h$ and g(x) = 2x. Now, when $h \to 0$, $z \to x$. Taking the limits as h goes to zero we obtain

$$\begin{aligned}
\lim_{h\to 0} \frac{F_1[g(x+h)] - F_1[g(x)]}{h} &= \lim_{h\to 0} \frac{\int_{2x}^{2x+2h} \sin(t^2)\, dt}{h} \\
&= F_1'(g(x)) \\
&= f_1(g(x))g'(x) \\
&= \sin[(2x)^2]D_x(2x) \\
&= 2\sin(4x^2)
\end{aligned}$$

And same for the second integral

$$\begin{aligned}
F_2(x+h) - F_2(x) &= \int_x^{x+h} \sin(t^2)\, dt = f_2(c)h \\
\iff \frac{F_2(x+h) - F_2(x)}{h} &= \frac{\int_x^{x+h} \sin(t^2)\, dt}{h} = f_2(c)
\end{aligned}$$

where $x < c < x+h$ and $c \to x$ as $h \to 0$. Taking the limits as h goes to zero

$$\begin{aligned}
\lim_{h\to 0} \frac{F_2(x+h) - F_2(x)}{h} &= \lim_{h\to 0} \frac{\int_x^{x+h} \sin(t^2)\, dt}{h} \\
&= F_2'(x) \\
&= f_2(x) \\
&= \sin(x^2)
\end{aligned}$$

Now, since we decomposed the original function into a difference of two new functions, the derivative of the original function is the difference of the derivatives of these two new functions:

$$F'(x) = F_1' - F_2' = \boxed{2\sin(4x^2) - \sin(x^2)}$$

Problem 3

Question: Find the limit

$$\lim_{x \to 0} \frac{\int_0^x e^{\sin t}}{x} \, dt \tag{1}$$

Hint: applying the fundamental theorem of calculus should lead to a difference quotient.

Solution (Method 1: L'Hôpital's rule): In the numerator we have

$$F(x) = \int_0^x e^{\sin t} \, dt$$

and because this is an integral course, in the denominator we can recognise

$$G(x) = x = \int_0^x dt$$

with

$$F'(x) = e^{\sin x}$$
$$G'(x) = 1$$

So (1) ends up looking like

$$\lim_{x \to 0} \frac{F(x)}{G(x)} \quad \iff \quad \lim_{x \to 0} \frac{\int_0^x e^{\sin t} \, dt}{\int_0^x dt}$$

Now, let's apply L'Hôpital's rule:

$$\lim_{x \to 0} \frac{F(x)}{G(x)} = \lim_{x \to 0} \frac{F'(x)}{G'(x)} = \lim_{x \to 0} \frac{e^{\sin x}}{1} = e^{\sin x}\Big|_{x=0} = \boxed{1}$$

Solution (Method 2: Difference quotient): In the numerator of (1) we have a function

$$F(x) = \int_0^x e^{\sin t} \, dt$$

And so, applying the first fundamental theorem of calculus the limit expression becomes

$$\lim_{x \to 0} \frac{\int_0^x e^{\sin t}}{x} \, dt \quad \iff \quad \lim_{x \to 0} \frac{F(x) - F(0)}{x} \quad \iff \quad \lim_{x \to 0} \frac{F(x) - F(0)}{x - 0}$$

Which we can recognise as the limit of a difference quotient. The intermediate value theorem for integrals tells us that there now exists some $0 < c < 0 + x$ for which the difference quotient equals $f(c)$, that is

$$\frac{F(x) - F(0)}{x} = f(c)$$

Now, because $c \in (0, 0 + x)$, as $x \to 0$, $c \to 0$, and so taking the limits we have

$$\lim_{x \to 0} \frac{F(x) - F(0)}{x} = \lim_{x \to 0} f(c) = f(x) = e^{\sin x}\Big|_{x=0} = e^0 = \boxed{1}$$

Problem 4

Question: What is the average value of the function

$$f(x) = \sqrt{2x}$$

on the interval $[0, 4]$? Since f is continuous it attains all of its values between its minimum and maximum on a closed interval. At which point does f attain its average value?

Solution (average value of the function): The arithmetic mean in the discrete case is defined as

$$\begin{aligned} \overline{a} &= \frac{a_1 + a_2 + ... + a_{n-1} + a_n}{n} \\ \iff \quad n\overline{a} &= a_1 + a_2 + ... + a_{n-1} + a_n \end{aligned}$$

We can apply this same logic for the continuous case. In the continuous case the number of observations over $[a, b]$ is its length, $b - a$. The sum of the values is an integral of some function over that interval. We arrive at

$$\begin{aligned} \overline{f} &= \frac{\int_0^4 \sqrt{2x}}{4 - 0} \\ &= \frac{1}{4} \int_0^4 \sqrt{2x} \\ &= \frac{1}{4} \int_0^4 \sqrt{2}\sqrt{x} \\ &= \frac{\sqrt{2}}{4} \int_0^4 x^{1/2} \\ &= \frac{\sqrt{2}}{4} \left[\frac{2}{3} (x)^{3/2} \right]_0^4 \\ &= \frac{\sqrt{2}}{4} \left[\frac{2}{3} (4)^{3/2} - \frac{2}{3} (0)^{3/2} \right] \\ &= \frac{\sqrt{2}}{4} \left[\frac{2}{3} \sqrt{4^3} \right] = \frac{\sqrt{2}}{6} \cdot \sqrt{64} = \frac{\sqrt{128}}{6} = \frac{\sqrt{2^6}\sqrt{2}}{6} = \frac{8\sqrt{2}}{6} = \boxed{\frac{4\sqrt{2}}{3}} \end{aligned}$$

Solution (the point at which f attains its average value): We are now looking for the value of x at which $f(x) = \sqrt{2x}$ attains the value $\frac{4\sqrt{2}}{3}$. Let's denote this value of x with c and let's solve it from

$$\overline{f} = f(c) = \sqrt{2c} = \frac{4\sqrt{2}}{3}$$

This can be done as

$$\begin{aligned} \sqrt{2c} = \sqrt{2}\sqrt{c} = \frac{4\sqrt{2}}{3} \\ \iff \quad \sqrt{c} = \frac{4}{3} \\ \iff \quad c = \left(\frac{4}{3}\right)^2 = \boxed{\frac{16}{9}} \end{aligned}$$

That is, $f(x) = \sqrt{2x}$ over $[0, 4]$ attains its average value $\frac{4\sqrt{2}}{3}$ at $x = \frac{16}{9}$.