Workshop 4: Integration by Substitution and by Parts

Problem 1

Question: Compute

$$\int \frac{dx}{x(x^p + a)}$$

where $a, p \neq 0$. Use the substitution $u = 1 + ax^{-p}$. Hint: First rewrite the expression to be integrated into a form in which the term $1 + ax^{-p}$ appears.

Solution: We have

$$\begin{aligned} \int \frac{dx}{x(x^p + a)} &= \int \frac{dx}{x^{1+p} + ax} \\ &= \int \frac{dx}{x^{1+p}(1 + ax^{-p})} \end{aligned}$$

Substitute $u = 1 + ax^{-p}$. Then $du = -pax^{-(1+p)} = -pa\frac{1}{x^{1+p}}\, dx$ and so

$$\begin{aligned} \int \frac{dx}{x^{1+p}(1 + ax^{-p})} &\iff \int \frac{dx}{x^{1+p}(u)} \\ &\iff -\frac{1}{pa}\int \frac{du}{u} \end{aligned}$$

and

$$-\frac{1}{pa}\int \frac{du}{u} = -\frac{1}{pa}\ln|u| + C = \boxed{-\frac{1}{pa}\ln|1 - ax^{-p}| + C, \quad a, p \neq 0}$$

Problem 2

Question: Using integration by substitution, show that

$$\int \frac{dx}{1 + 3\cos^2 x} = \frac{1}{2}\arctan\left(\frac{\tan x}{2}\right) + C$$

Use the substitution $u = \frac{\tan x}{2}$ and recall that $\cos^{-2} x = \tan^2 x + 1$

Solution: Let's multiply the numerator and the denominator by $1/\cos^2 x = \sec^2 x$:

$$\begin{aligned}\int \frac{dx}{1+3\cos^2 x} &= \int \frac{1/\cos x}{\frac{1}{\cos^2 x}+3}\,dx \\ &= \int \frac{\sec^2 x}{\sec^2 x+3}\,dx \\ &= \int \frac{\sec^2 x}{(\tan^2 x+1)+3}\,dx && \text{Trig identity: } (\sec^2 x = \tan^2 x + 1) \\ &= \int \frac{1}{4+\tan^2 x}\sec^2 x\,dx \\ &= \int \frac{1}{4}\cdot\frac{1}{1+\frac{\tan^2 x}{4}}\sec^2 x\,dx \\ &= \frac{1}{2}\int \left[\frac{1}{1+(\frac{\tan x}{2})^2}\right]\cdot\left[\frac{1}{2}\sec^2 x\,dx\right]\end{aligned}$$

Let's now substitute $u = \frac{\tan x}{2}$. Then $du = \frac{1}{2}\cdot\sec^2 x\,dx$ because

$$\begin{aligned}\frac{du}{dx}\left[\frac{1}{2}\tan x\right] &= \frac{1}{2}\frac{du}{dx}\left[\frac{\sin x}{\cos x}\right] \\ &= \frac{1}{2}\left[\frac{D\ \sin x\cdot\cos x - \sin x\cdot D\ \cos x}{(\cos x)^2}\right] \\ &= \frac{1}{2}\cdot\frac{\cos^2 x+\sin^2 x}{\cos^2 x} \\ &= \frac{1}{2}\cdot\frac{1}{\cos^2 x} && \text{Trig identity: } (\sin^2+\cos^2 = 1) \\ &= \frac{1}{2}\cdot\sec^2 x\end{aligned}$$

Applying the substitution, we obtain

$$\frac{1}{2}\int \frac{1}{1+u^2}\,du$$

We know that $D_x \arctan x = \frac{1}{1+x^2}$, and so

$$\boxed{\frac{1}{2}\int \frac{1}{1+u^2}\,du \quad = \quad \frac{1}{2}\arctan u + C \quad = \quad \frac{1}{2}\arctan\left(\frac{\tan x}{2}\right)+C}$$

Problem 3

Question: Compute

$$\int x^4 e^{-x}\,dx$$

using the reduction formula from Example 2:

$$I_n = -x^n e^{-x} + nI_{n-1}, \quad n \geq 0$$

Solution: The reduction formula gives

$$\begin{aligned}
I_4 &= -x^4e^{-x} + 4I_3 \\
I_3 &= -x^3e^{-x} + 3I_2 \\
I_2 &= -x^2e^{-x} + 2I_1 \\
I_1 &= -xe^{-x} + I_0 \\
I_0 &= -e^{-x}
\end{aligned}$$

Therefore

$$I_4 = \int x^4e^{-x}\,dx = \boxed{-e^{-x}\Big(x^4 + 4x^3 + 12x^2 + 24x + 24\Big)}$$

Problem 4

Question: Use integration by parts to compute

$$\int \sin\big(\ln x\big)\,dx$$

Hint: In order to integrate by parts, you will need to rewrite the expression as a product of two functions one of which is very simple. You will also need to use this trick twice. The computation will eventually lead to an expression in which the original integral appears. That allows writing an expression for it without integrals.

Solution: Let's multiply the integral with $\frac{x}{x} = 1$. We obtain

$$\int x\sin\big(\ln x\big)\frac{1}{x}\,dx$$

Now we have an integral in the form

$$\int f(x)g'(h(x))h'(x)$$

which we can integrate by parts:

$$\int f(x)g'(h(x))h'(x)\,dx = f(x)\int g'(h(x))h'(x)\,dx - \int f'(x)g(h(x))\,dx$$

that is,

$$\begin{aligned}
\int x\sin\big(\ln x\big)\frac{1}{x}\,dx &= x\int \sin\big(\ln x\big)\frac{1}{x}\,dx + \int 1\cdot\cos(\ln x)\,dx \\
&= -x\cos(\ln x) + \int \cos(\ln x)\,dx
\end{aligned}$$

Let's use the same trick again on the remaining integral; let's multiply by $\frac{x}{x} = 1$. Then

$$\int \cos(\ln x)\,dx = \int x\cos(\ln x)\frac{1}{x}\,dx = x\sin(\ln x) - \int \sin(\ln x)\,dx$$

Collecting the calculations together, we have

$$\begin{aligned}\int \sin(\ln x)\,dx &= -x\cos(\ln x) + \int \cos(\ln x)\,d\\ &= -x\cos(\ln x) + x\sin(\ln x) - \int \sin(\ln x)\,dx\\ &\iff 2\int \sin(\ln x)\,dx = -x\cos(\ln x) + x\sin(\ln x)\\ &\iff \int \sin(\ln x)\,dx = \boxed{-\frac{x}{2}\Big[\cos(\ln x) - \sin(\ln x)\Big]}\end{aligned}$$

Problem 5

Question: Compute

$$\int \sin(hx)\cos x\,dx$$

Hint: Use integration by parts twice.

Solution:

$$\begin{aligned}\int \sin(hx)\cos x\,dx &= \sin(hx)\sin x - \int h\cos(hx)\cos x\\ &= \sin(hx)\sin x - \Big[-h\sin x\cos(hx) + \int h^2\sin(hx)\cos x\Big]\\ &= \sin(hx)\sin x + h\sin x\cos(hx) - \int h^2\sin(hx)\cos x\end{aligned}$$

Now again we have the original integral appearing on the RHS, this time scaled by h^2. Solving for it we obtain

$$\begin{aligned}(1+h^2)\int \sin(hx)\cos x\,dx &= \sin(hx)\sin x + h\sin x\cos(hx)\\ \iff \int \sin(hx)\cos x\,dx &= \frac{1}{1+h^2}\Big[\sin(hx)\sin x + h\sin x\cos(hx)\Big]\\ &= \boxed{\frac{\sin x}{1+h^2}\Big[\sin(hx) + h\cos(hx)\Big]}\end{aligned}$$