Macroeconomics 2: Problem set 2

Problem 1

Part a)

Question: Assume that there are three wage income realisations (s^1, s^2, s^3), and the probability transition matrix is

$$P = \begin{bmatrix} 1-\pi_1 & \pi_1 & 0 \\ 0.5\pi_2 & 1-\pi_2 & 0.5\pi_2 \\ 0 & \pi_3 & 1-\pi_3 \end{bmatrix}$$

1. Find the stationary wage distribution λ_s, that is, express λ_s^i in terms of π_1, π_2, π_3.
2. Assume further that $\pi_1 = \pi$, $\pi_2 = 0.5\pi$ and $\pi_3 = 0.25\pi$ where $\pi \in (0, 1)$. What is the wage distribution? What happens to the distribution when π changes? How would you interpret your finding?

Solution 1 (stationary wage distribution): We have a Markov chain, and so we can express the distribution for the next state for all three states:

$$\begin{aligned} \lambda_{t+1}^1 &= (1-\pi_1)\lambda_t^1 + 0.5\pi_2\lambda_t^2 \\ \lambda_{t+1}^2 &= \pi_1\lambda_t^1 + (1-\pi_2)\lambda_t^2 + \pi_3\lambda_t^3 \\ \lambda_{t+1}^3 &= 0.5\pi_2\lambda_t^2 + (1-\pi_3)\lambda_t^3 \end{aligned} \tag{1}$$

Next we assume the condition for stationarity

$$\lambda_{t+1}^n = \lambda_t^n \overset{def}{=} \lambda_s^n$$

holds, and we substitute the stationary probability into (1):

$$\begin{aligned} \lambda_s^1 &= (1-\pi_1)\lambda_s^1 + 0.5\pi_2\lambda_s^2 \\ \lambda_s^2 &= \pi_1\lambda_s^1 + (1-\pi_2)\lambda_s^2 + \pi_3\lambda_s^3 \\ \lambda_s^3 &= 0.5\pi_2\lambda_s^2 + (1-\pi_3)\lambda_s^3 \end{aligned}$$

these, in addition to the condition $\lambda_s^1 + \lambda_s^2 + \lambda_s^3 = 1$, allow us to solve for the stationary probabilities:

$$\lambda_s^1 = \frac{\pi_2\pi_3}{2\pi_1\pi_3 + \pi_1\pi_2 + \pi_2\pi_3}$$
$$\lambda_s^2 = \frac{2\pi_1\pi_3}{2\pi_2\pi_3 + 2\pi_1\pi_3 + \pi_1\pi_2 - \pi_2\pi_3}$$
$$\lambda_s^3 = \frac{\pi_1\pi_2}{2\pi_1\pi_3 + \pi_1\pi_2 + \pi_2\pi_3}$$

Solution 1 (further assumptions): Let's assume further that

$$\pi_1 = \pi$$
$$\pi_2 = \frac{1}{2}\pi$$
$$\pi_3 = \frac{1}{4}\pi$$

Now the stationary distribution becomes

$$\lambda_s^1 = \frac{\pi_2\pi_3}{2\pi_1\pi_3 + \pi_1\pi_2 + \pi_2\pi_3} = \frac{1}{9}$$
$$\lambda_s^2 = \frac{2\pi_1\pi_3 + \pi_1\pi_2 - \pi_3\pi_2}{2\pi_3} = \frac{4}{9}$$
$$\lambda_s^3 = \frac{\pi_1\pi_2}{2\pi_1\pi_3 + \pi_1\pi_2 + \pi_2\pi_3} = \frac{4}{9}$$

And so the wage distribution is

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \end{bmatrix}$$

which means it does not change with the value of π.

Part b)

Question: Assume that we have 5 different wage income levels and that the probability transition matrix is

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} \\ p_{41} & p_{42} & p_{43} & p_{44} & p_{45} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $p_{ij} \in (0, 1)$ for all $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$. What is the stationary wage distribution? (Note: you do not have to do any calculations to answer this question. Just inspect the transition matrix).

Solution: The stationary wage distribution is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

because state 4 (5 if you start indexing from 1) is an absorbing state. Because $0 < p_{ij} < 1$, the process will end up in state 4 with probability 1 as $t \to \infty$.

Part c)

Question: Assume that there are 4 different income levels and

1. The transition matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & p_{22} & p_{23} & p_{24} \\ 0 & p_{32} & p_{33} & p_{34} \\ 0 & p_{42} & p_{43} & p_{44} \end{bmatrix}$$

   where $p_{ij} \in (0,1)$ for all $i, j = 2, 3, 4$

2. The transition matrix is

$$P = \begin{bmatrix} p_{11} & p_{12} & 0 & 0 \\ p_{21} & p_{22} & 0 & 0 \\ 0 & 0 & p_{33} & p_{34} \\ 0 & 0 & p_{43} & p_{44} \end{bmatrix}$$

   where $p_{ij} \in (0,1)$ and $p_{mn} \in (0,1)$ for $i, j = 1, 2$ and $m, n = 3, 4$

Explain why there does not exist a unique stationary distribution in the situations captured by matrices (3) and (4). How would you interpret the situations described by matrices (3) and (4)?

Solution: There are no unique stationary distributions for the matrices in question because both have two communication classes which are both absorbing classes: if the process ends up in either class, it will never leave it. The stationary distribution then depends on the initial distribution.

The interpretation is that there exists some irreversible point of no return, after which the dynamics change permanently.

Part d)

Question: Assume that there are two wage shock realisations (s^1 and s^2) and the shock transition matrix is

$$P = \begin{bmatrix} 1 - \pi_1 & \pi_1 \\ \pi_2 & 1 - \pi_2 \end{bmatrix}$$

We assume that the consumer can choose between three wealth levels $a^0 = 0$, $a^1 = 1$ and $a^2 = 2$. Also assume that the policy function $g(a, s)$, expressing future assets as a function of current assets and current wage, is given by

$g(a,s)$	a^0	a^1	a^2
s^1	0	0	0
s^2	1	2	2

Write down the transition matrix Q for joint and wage wealth dynamics.

Solution: Using the indicator function we arrive at the following wealth transition matrices. For $s = s^1$:

$$\mathcal{I}_{s^1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and for $s = s^2$:

$$\mathcal{I}_{s^2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we can compute the joint wealth/wage dynamics transition matrix

$$\mathcal{Q} = \begin{bmatrix} (1-\pi_1)\mathcal{I}_{s^1} & \pi_1 \mathcal{I}_{s^1} \\ \pi_2 \mathcal{I}_{s^2} & (1-\pi_2)\mathcal{I}_{s^2} \end{bmatrix}$$

Problem 2

Part a)

Question: Your task is to find the stationary wealth distribution of this economy. (Hint: The distribution is of a very simple form, and you essentially only need the threshold wealth level n to characterize it. Also notice that the shocks are independently and identically distributed. This makes finding the wealth distribution easier.

Solution: The probabilistic flow budget constraints separately for consumers above and below the threshold, respectively, are

$$\begin{aligned} a' &= a + 1 - 2 \\ &= a - 1 \end{aligned}$$

and

$$\begin{aligned} a' &= a + 1 - 2 \cdot \frac{1}{2} - 0 \cdot \frac{1}{2} \\ &= a' \end{aligned}$$

That is, consumers below the threshold are expected to stay where they are, while those above the threshold are expected to converge to the threshold level of wealth.

This means that the stationary wealth distribution is uniform between 1 and n^* for p percent of the population, while the rest of the population, $1-p$, will have wealth n^*.

Part b)

Question: The consumers are of mass unity, and in each period the total supply of the consumption good is also of mass unity. Remember that everyone who consumes has to consume two units. Does the goods market clear? If yes, what equilibrium price clears the market?

Solution: If n consumers at all times were demanding two goods when supply was n, the price of a good would become

$$\frac{p}{2} = \frac{n}{n} = 1 \quad \Longrightarrow \quad p = 2$$

But in this set up, not all consumers are consuming at all times. Say $h \in [0, 1]$ of all consumers participate in the long run, on average. Then the long run price will be

$$\frac{p}{2} = \frac{hn}{n} = h \quad \Longrightarrow \quad p = 2h$$

Problem 3

Consumer maximises

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$a' + c = (1 + r)a + s$$

The utility function is

$$u(c) = \ln c$$

The beta is

$$\beta = 0.95 = 19/20$$

The probability transition matrix being

$$P = \begin{bmatrix} 2/3 & 1/3 \\ 1/2 & 1/2 \end{bmatrix}$$

Wealth grid has three points

$$A^d = \{a^0, a^1, a^2\} = \{0, 1, 2\}$$

The stochastic process takes on two values

$$s \in S = \{s^1, s^2\} = \{1, 2\}$$

Question: Conduct two rounds of value function iteration and report the value function $V^2(a, s)$, the policy function $a^2 = g^2(a, s)$, and consumption $c^2(a, s)$ after the second round. You should also document the procedure you apply.

Solution: Denote

$$V(a,s) = V(a_0, s_0) = V_0(a,s) = \max_{a'} \left\{ \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t) \right\}$$

and

$$V(a_1, s_1) = V_1(a,s) = \max_{a'} \left\{ \mathbb{E} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \right\}$$

Then we can write the problem recursively as

$$\begin{aligned} V(a,s) &= \max_{a'} \left\{ u(c) + \beta \mathbb{E}_{s'|s} V(a', s') \right\} \\ &= \max_{a'} \left\{ \ln c + \beta \mathbb{E}_{s'|s} V(a', s') \right\} \end{aligned}$$

We have an **initial guess** of $V_0(a,s) = 0$. Let's proceed to substitute it into the above equation:

$$\begin{aligned} V_1(a,s) &= \max_{a'} \left\{ \ln c + \beta \mathbb{E}_{s'|s} V_0(a', s') \right\} \\ &= \max_{a'} \left\{ \ln c \right\} \end{aligned}$$

For consumption we have the tables

$s=1$	$a'=0$	$a'=1$	$a'=2$
$a=0$	1	0	-1
$a=1$	2	1	0
$a=2$	3	2	1

and

$s=3$	$a'=0$	$a'=1$	$a'=2$
$a=0$	3	2	1
$a=1$	4	3	2
$a=2$	5	4	3

and so for utility we have the tables

$s=1$	$a'=0$	$a'=1$	$a'=2$
$a=0$	$\ln c = 0$	$\ln c = -\infty$	$\ln c = undefined$
$a=1$	$\ln c = 0.69$	$\ln c = 0$	$\ln c = -\infty$
$a=2$	$\ln c = 1.1$	$\ln c = 0.69$	$\ln c = 0$

and

$s=3$	$a'=0$	$a'=1$	$a'=2$
$a=0$	$\ln c = 1.1$	$\ln c = 0.69$	$\ln c = 0$
$a=1$	$\ln c = 1.39$	$\ln c = 1.1$	$\ln c = 0.69$
$a=2$	$\ln c = 1.61$	$\ln c = 1.39$	$\ln c = 1.1$

We start with an **initial guess** of $V_0(a,s) = 0$ and thus obtain the table

$V_0(a,s)$	$a'=0$	$a'=1$	$a'=2$
$s'=1$	0	0	0
$s'=3$	0	0	0

Then the expected continuation value is

$$\beta \mathbb{E}_{s'|s} V_0(a', s') = \beta \sum P(s'|s) V_0(a', s')$$

and in tabular form

$\beta \mathbb{E}_{s'\|s} V_0(a', s')$	$a'=0$	$a'=1$	$a'=2$
$s'=1$	0	0	0
$s'=3$	0	0	0

First iteration:

$V_1(a,s)$	$a=0$	$a=1$	$a=2$
$s=1$	0	0.69	1.1
$s=3$	1.1	1.39	1.61

$c_1(a,s)$	$a=0$	$a=1$	$a=2$
$s=1$	1	2	3
$s=3$	1	2	3

$a'_1(a,s)$	$a=0$	$a=1$	$a=2$
$s=1$	0	0	0
$s=3$	0	0	0

Process: when $a = 0, s = 1$, consumers chooses $a' = 0$ because $0 > -\infty$. Then $V_1(a,s) = 0$ and $c = a + 1 - a' = 1$. Repeat for other cases.

Second iteration:
Now we have

$$V_2(a,s) = \max_{a'} \left\{ \ln c + \beta \mathbb{E}_{s'|s} V_1(a', s') \right\}$$

The expected continuation value is

$$\beta \mathbb{E}_{s'|s} V_1(a', s') = \beta \sum P(s'|s) V_1(a', s') \Longleftrightarrow \begin{bmatrix} 2/3 & 1/3 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 0.69 & 1.1 \\ 1.1 & 1.39 & 1.61 \end{bmatrix}$$
$$\Longrightarrow \begin{bmatrix} 0.37 & 0.92 & 1.27 \\ 0.55 & 1.04 & 1.36 \end{bmatrix}$$

And in tabular form

$\beta \mathbb{E}_{s'\|s} V_0(a', s')$	$a' = 0$	$a' = 1$	$a' = 2$
$s' = 1$	0.37	0.92	1.27
$s' = 3$	0.55	1.04	1.36

Now we obtain the tables for the second iteration using the process outline above:

$V_1(a, s)$	$a = 0$	$a = 1$	$a = 2$
$s = 1$	0.69	1.1	1.1
$s = 3$	1.61	1.61	1.61

$c_1(a, s)$	$a = 0$	$a = 1$	$a = 2$
$s = 1$	0	0	1
$s = 3$	1	2	3

$a'_1(a, s)$	$a = 0$	$a = 1$	$a = 2$
$s = 1$	1	2	2
$s = 3$	2	2	2

which concludes the two rounds of value function iteration.