Integrating like a statistician

Take the constant function

$$g(x) = 1$$

And integrate over $[a, b]$

$$\int_a^b dx = \Big[x\Big]_a^b = (b - a) \neq 1$$

$\implies$ Not a pdf! But $\frac{b-a}{b-a} = 1$, so scale $g(x)$ by $\frac{1}{b-a}$:

$$f(x) = \frac{1}{b-a} g(x) = \frac{1}{b-a}$$

and integrate over $[a, b]$

$$\frac{1}{b-a} \int_a^b dx = \frac{1}{b-a} \Big[x\Big]_a^b = \frac{b-a}{b-a} = 1$$

So $f(x) = \frac{1}{b-a}$ over $[a, b]$ **is** a pdf.

Scaling variance / expectation value

The property of variance

$$\begin{aligned}
\mathbb{V}[aX] &= \mathbb{E}[(aX - \mathbb{E}[aX])(aX - \mathbb{E}[aX])] \\
&= \mathbb{E}[(aX - a\,\mathbb{E}[X])(aX - a\,\mathbb{E}[X])] \\
&= \mathbb{E}[a(X - \mathbb{E}[X])a(X - \mathbb{E}[X])] \\
&= \mathbb{E}[a^2(X - \mathbb{E}[X])(X - \mathbb{E}[X])] \\
&= a^2\,\mathbb{E}[(X - \mathbb{E}[X])(X - \mathbb{E}[X])] \\
&= a^2\,\mathbb{V}[X]
\end{aligned}$$

means we can manipulate the parameters defining pdf's of random variables. For example, if $X \sim N(0, \sigma^2)$, then

$$\mathbb{V}\,X = \mathbb{E}[(X - \mathbb{E}\,X)^2] = \sigma^2$$

And if we now want the variance to equal 1, we can work backwards to find a scaling magnitude that achieves that

$$\begin{aligned}
1 &= \frac{\sigma^2}{\sigma^2} \\
&= \frac{1}{\sigma^2}\sigma^2 \\
&= \frac{1}{\sigma^2}\,\mathbb{E}[(X - \mathbb{E}\,X)^2] \\
&= \mathbb{E}[(\frac{1}{\sigma^2})(X - \mathbb{E}\,X)^2] \\
&= \mathbb{E}[(\frac{1}{\sigma})(X - \mathbb{E}\,X)(\frac{1}{\sigma})(X - \mathbb{E}\,X)] \\
&= \mathbb{E}[(\frac{1}{\sigma}X - \frac{1}{\sigma}\,\mathbb{E}\,X)(\frac{1}{\sigma}X - \frac{1}{\sigma}\,\mathbb{E}\,X)] \\
&= \mathbb{E}[(\frac{1}{\sigma}X - \mathbb{E}\,\frac{1}{\sigma}X)(\frac{1}{\sigma}X - \mathbb{E}\,\frac{1}{\sigma}X)] \\
&= \mathbb{V}[\frac{1}{\sigma}X]
\end{aligned}$$

So scaling X by $\frac{1}{\sigma}$ makes the resulting random variable to have a variance of one.

Scaling variance with matrix notation

Let $\mathbb{V}\,\boldsymbol{x} = \boldsymbol{\Omega}$. Then

$$\begin{aligned}
\mathbb{V}[\boldsymbol{Ax}] &= \mathbb{E}[(\boldsymbol{Ax} - \mathbb{E}[\boldsymbol{Ax}])(\boldsymbol{Ax} - \mathbb{E}[\boldsymbol{Ax}])'] \\
&= \mathbb{E}[(\boldsymbol{Ax} - \boldsymbol{A}\,\mathbb{E}[\boldsymbol{x}])(\boldsymbol{Ax} - \mathbb{E}[\boldsymbol{x}]\boldsymbol{A})'] \\
&= \mathbb{E}[\boldsymbol{A}(\boldsymbol{x} - \mathbb{E}[\boldsymbol{x}])(\boldsymbol{x} - \mathbb{E}[\boldsymbol{x}])'\boldsymbol{A}'] \\
&= \boldsymbol{A}\,\mathbb{E}[(\boldsymbol{x} - \mathbb{E}[\boldsymbol{x}])(\boldsymbol{x} - \mathbb{E}[\boldsymbol{x}])']\boldsymbol{A}' \\
&= \boldsymbol{A}\,\mathbb{V}\,\boldsymbol{x}\boldsymbol{A}' \\
&= \boldsymbol{A\Omega A}'
\end{aligned}$$